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HOMOLOGY OF SOME GROUPS OF PL-HOMEOMORPHISMS OF THE LINE

Peter GREENBERG
Matematicas, Cinvestav, Apartado Postal 14-740, Mexico 14 DF, CP 07000, Mexico

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If G is a discrete group, the homology $H_*(G)$ of G is defined to be the integral homology of a $K(G,1)$, or BG In earlier work we showed how to calculate the homology groups of groups of orientation preserving, piecewise linear (pl-) homeomorphisms of the line $\mathbb{R}$, whose breakpoints (points of discontinuity of the derivative) are isolated Here, we extend the calculation to groups of pl-homeomorphisms with more complicated sets of breakpoints

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classifying spaces   pseudogroups
homology of groups

Introduction

Namely, if $S\subseteq\mathbb{R}$ let $I(S)=\{s\in S.\ \exists\varepsilon$ such that $\forall s'\in S-s, |s-s'|>\varepsilon\}$, and let $L^1(S)=L(S)=S-I(S)$ be the (first) derived set of S. A set is discrete if $L(S)$ is empty. Let $L^n(S)=L^{n-1}(L(S))$ be the nth derived set of S. Let K^n be the group of compactly supported homeomorphisms $g:\mathbb{R}\to\mathbb{R}$, such that there exists some $S\subseteq\mathbb{R}$, with $L^n(S)=\emptyset$, such that for each component I of $\mathbb{R}-S$ there exist $b_I\in\mathbb{R}$, $a_I\in\mathbb{R}^+$ so that $g(x)=a_Ix+b_I$, $x\in I$ The following follows immediately from Corollary 1.6 and [3]

0.1. Example. (a) [3, 4] $H_*(K^1)=H_*(\Omega B\Gamma^1)$ where $B\Gamma^1$ is weakly (homotopy) equivalent to $B\mathbb{R}^+ * B\mathbb{R}^+$.

(b) $H_*(K^2)=H_*(\Omega B\Gamma^2)$ where $B\Gamma^2$ is weakly equivalent to $(B\Gamma^1 * B\Gamma^1)\vee SB\Gamma^1\vee S^2(B\mathbb{R}^+\times B\mathbb{R}^+)$.

(c) For $n\geqslant 2$, $H_*(K^{n+1})=H_*(\Omega B\Gamma^{n+1})$ where $B\Gamma^{n+2}$ is weakly equivalent to $(B\Gamma^n * B\Gamma^n)\vee SB\Gamma^n\vee(SB\Gamma^{n-1} * B\Gamma^{n-1})\vee 2S^2B\Gamma^{n-1}$

Here, Ω and S are loops and suspension, $\mathbb{R}^+$ is the discrete group of positive reals, $*$ and $\vee$ are join and wedge, and $2X$ means $X\vee X$. The significance of the

$B\Gamma^n$ is discussed more fully in Section 1 A "smaller" and perhaps more interesting example is Example 1 9

0.2. Corollary. $H_* K^n = 0, * < n$

In [8], Mather proved the acyclicity of the group G of compactly supported homeomorphisms of the line The crucial construction is as follows. Given $g \in G$ with support in some interval I, there exists $\psi(g) \in G$, with support in an infinite union of disjoint intervals $I \cup I_1 \cup I_2 \cup \cdots$, such that $\psi(g)|_I \equiv g$, $\psi(g)|_{I_n}$ is conjugate to g Further, if $\varphi(g) = g^{-1}\psi(g)$, then φ and ψ are conjugate in G so that $g \equiv 0$ in $H_1 G$ The construction fails for the groups K^n above, because the I_k must have a limit point in order that $\psi(g)$ be compactly supported. Thus $\psi(g) \in K^{n+1}$, and the following consequence of Corollary 1 6 is now reasonable

0.3. Corollary. $K^n \to K^{n+1}$ *induces the zero map on homology*, $n \geq 1$

In [1] (see also [7, 14]) it is shown how to enlarge a group K to a group mK, so that the inclusion of K into mK is zero on homology For certain groups K of homeomorphisms of the line, we do this geometrically, by adding singularities, to obtain a sequence of groups of homeomorphisms $K \subseteq K^1 \subseteq K^2 \subseteq \cdots$ so that $K^n \to K^{n+1}$ is zero in homology Further (Corollary 1.7) $H_* K^n = 0, * < n$ Indeed, there is a recursion formula, as in Example 0.1(c), for the homology groups of the K^i

As in [3, 4] our method is to deal with pseudogroups on $\mathbb{R}$, and their classifying spaces $B\Gamma$ Thus, the groups for which our theorems apply are groups of homeomorphisms of the line, and groups of compactly supported homeomorphisms of the line, associated to "sufficiently nonanalytic" (Assumptions 1.1) pseudogroups

1. Construction and results

Let Γ be a pseudogroup of orientation preserving homeomorphisms of the real line $\mathbb{R}$ That is [5], Γ is a collection of homeomorphisms between open subsets of the real line, which is closed under restriction to open sets, inverse, and, where possible, composition. Associated to Γ is the group H_Γ of orientation preserving elements of Γ whose domain and range are the whole real line, and its subgroup K_Γ of compactly supported homeomorphisms in Γ. There is a classifying space functor $\Gamma \to B\Gamma$ whose utility, for the present work, is that if Γ is sufficiently nonanalytic ("germ-connected to the identity", Definition 1 10, see [9, 11, 4]) there are maps $BK_\Gamma \to \Omega B\Gamma$, $BH_\Gamma \to B\Gamma$ inducing isomorphism in homology. So we will "add singularities" to pseudogroups.

1.1. Assumptions on Γ. From now on we will assume that there is a point $B \in \mathbb{R}$ so that the orbit $\Gamma(B)$ of B under Γ is dense, and we will assume chosen such a *basepoint* B when Γ is given. The homotopy type of our constructions will only depend on the orbit chosen, not on the point B. We further assume that, given $a<b$, $c<d$ in $\Gamma(B)$, there exist $x_i \in \Gamma(B)$, $a=x_0<x_1<\cdots<x_n<x_{n+1}=b$ and $g_i \in \Gamma$, $0 \leq i \leq n$ so that (see Fig. 1)

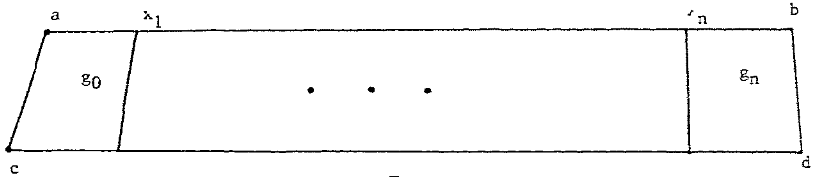

Fig 1

(i) $[x_i, x_{i+1}]$ is in the domain of g_i and $g_i(x_{i+1}) = g_{i+1}(x_{i+1})$,
(ii) $g_0(a)=c$, $g_n(b)=d$.
(See [4].)

We now define, beginning with $\Gamma = \Gamma^0$, a sequence of pseudogroups $\Gamma^0 \subset \Gamma^1 \subset \cdots$.

1.2. Definition. Let $S \subseteq \mathbb{R}$. Then

$$I(S) = \{s \in S: \text{there exists } \varepsilon > 0, (s-\varepsilon, s+\varepsilon) \cap S = s\}$$

is the set of *isolated points* of S. Let

$$L^1(S) = S - I(S). \qquad L^n(S) = L^1(L^{n-1}S), \quad n \geq 2.$$

1.3. Definition. Let Γ be a pseudogroup of orientation preserving homeomorphisms between open subsets of the real line with basepoint B. Then Γ^n is the pseudogroup of homeomorphisms $g: U \to V$ between open subsets of $\mathbb{R}$, such that there exists $S \subseteq \Gamma(B) \cap U$ such that $L^n(S) = \emptyset$, and such that for each component I of $U - S$, there exists $g_I \in \Gamma$ with $g|_I \equiv g_I$.

It is straightforward that the Γ^n are pseudogroups, and that $\Gamma^n \subseteq \Gamma^{n+1}$ for all $n \geq 0$. Let G be the group of Γ-germs at B, and let G^1 be the group of Γ^1-germs at B.

1.4. Theorem. (i) *$B\Gamma^n \to B\Gamma^{n+1}$ is nullhomotopic, $n \geq 1$.*

(ii) *If $n \geq 2$, $B\Gamma^{n+1}$ is weakly (homotopy) equivalent to $(B\Gamma^n * B\Gamma^n) \vee SB\Gamma^n \vee (SB\Gamma^{n-1} * B\Gamma^{n-1}) \vee 2S^2B\Gamma^{n-1}$.*

(iii) *$B\Gamma^2$ is weakly equivalent to $(B\Gamma^1 * B\Gamma^1) \vee SB\Gamma^1 \vee S^2(BG^1)$.*

Here, $*$ is the join, $\vee$ is the wedge, and S is suspension. If X is a space, $2X$ means $X \vee X$.

By [4, 1.12], if Γ satisfies Assumptions 1.1 then Γ^1, and hence all the Γ^n, $n \geq 1$, are germ-connected to the identity [4, 1.4] which implies the following [4, 1.5] but essentially due to Segal [11] and Mather [9]:

1.5. Fact. *Let $K^n \subset H^n$, $n \geq 0$, be the groups of compactly supported homeomorphisms and global homeomorphisms associated to the Γ^n. Then for $n \geq 1$ we have that $B\Gamma^n$ is simply connected, and the maps $BH^n \to B\Gamma^n$, $BK^n \to \Omega B\Gamma^n$ induce isomorphisms in homology.*

1.6. Corollary. (i) *For $n \geq 1$ the inclusions $BH^n \to BH^{n+1}$, $BK^n \to BK^{n+1}$ are null in homology.*

(ii) *For $n \geq 2$, BH^{n+1} has the homology of $(BH^n * BH^n) \vee SBH^n \vee (SBH^{n-1} * BH^{n-1}) \vee 2SBH^{n-1}$, and BH^2 has the homology of $(BH^1 * BH^1) \vee SBH^1 \vee S^2(BG^1)$.*

(iii) *For $n \geq 1$, BK^n has the homology Hopf algebra of $\Omega B\Gamma^n$.*

1.7. Corollary. *$B\Gamma^n$ is n-connected, $H_k(H^n) = 0$, $1 \leq k \leq n$, and $H_k(K^n) = 0$, $1 \leq k \leq n-1$. Furthermore, let $\Gamma = \bigcup \Gamma^n$, $H = H_I = \bigcup H^n$ and $K = K_I = \bigcup K^n$. Then $B\Gamma$ is contractible, and H and K are acyclic.*

Proof. The first part follows because suspension and join raise connectivity. Note that the acyclicity of K is essentially proved in [8]. □

1.8. Remark. By Lemma 1.4, join distributes over the wedge. Thus, for spaces constructed from wedges and joins, the usual rules for polynomials apply. The suspension S acts like a free variable, because suspension is the same as join with an S^0. As a consequence, we can derive formulas for the $B\Gamma^n$ in terms of $B\Gamma^1$ and BG^1. Writing, for a space X, $X^n = X * \cdots * X$ (n times) and $nX = X \vee \cdots \vee X$ (n times), we have, for example, that $B\Gamma^3$ is weakly equivalent to $(B\Gamma^1)^4 \vee S^2(B\Gamma^1)^2 \vee S^4(BG^1)^2 \vee 2S(B\Gamma^1)^3 \vee 2S^2(B\Gamma^1)^2 * BG^1 \vee 2S^3B\Gamma^1 * BG^1 \vee 2S(B\Gamma^1)^2 \vee 3S^2B\Gamma^1 \vee S^3BG^1$. I have not found a closed form for $B\Gamma^n$, not even for the following example.

1.9. Basic example. Let A be the group of orientation preserving affine transformations of the ring $\mathbb{Z}[\frac{1}{2}]$. As a group acting on the real line, A consists of transformations $x \to 2^n x + a/2^m$, $n, m, a \in \mathbb{Z}$. Let Γ be the pseudogroup of restrictions of elements of A to open subsets of the real line, and set the basepoint $B = 0$. Then $\Gamma(B) = \mathbb{Z}[\frac{1}{2}]$ is dense in $\mathbb{R}$. It turns out (see [4]) that Γ satisfies Assumptions 1.1. Further, $B\Gamma$ is a $K(A, 1)$. By the definition, Γ^1 is the pseudogroup of piecewise linear homeomorphisms between open subsets of the line, preserving orientation, with derivative an integral power of 2 where defined, and with points of discontinuity of the derivative isolated points in $\mathbb{Z}[\frac{1}{2}]$. In [3] it is shown that $B\Gamma^1$ is weakly equivalent to S^3. Note that G^1 is isomorphic to $\mathbb{Z}^2$.

Now using Theorem 1.4, we find that $B\Gamma^2$ is weakly equivalent to $S^7 \vee S^4 \vee S^2(S^1 \times S^1)$, or, since suspension splits the torus, $S^7 \vee 2S^4 \vee 2S^3$. Thereafter, the $B\Gamma^{n+1}$ are wedges of spheres $2S^{n+2} \vee \cdots \vee B(k, n)S^k \vee \cdots \vee S^{a(n)}$, with $a(n) = 2^{n+2} - 1$. The coefficients of the intermediate spheres follow the formula

$$B(k,n) = B(k-1,n-1) + \sum_{j=0}^{k-2} B(j,n-2)B(k-2-j,n-2)$$
$$+ \sum_{j=1}^{k-2} B(j,n-1)B(k-1-j,n-1)$$

and the Pontrjagin rings $H_*(K^{n+1})$ are polynomial algebras with $B(k,n)$ generators in dimension $k-1$.

The universal Γ-structure on S^3 may be taken to be the Reeb foliation (see [3]). Are there nonsingular generalizations on the $S^{a(n)}$?

If we begin with a flabbier pseudogroup than the Γ above, we get a slightly stronger version of Theorem 1.4. Here flabby means

1.10. Definition. A pseudogroup Γ of orientation preserving homeomorphisms between open subsets of the real line is *germ-connected to the identity* if, given $g \in \Gamma$, $x \in \mathrm{domain}(g)$, and $\varepsilon > 0$, then if $g(x) \geq x$ (respectively $\leq x$) there exists an $s \in \Gamma$, whose domain is connected and contains $(-\infty, x]$ (respectively $[x, \infty)$) such that $s \equiv g$ near x, and such that for all $y < x - \varepsilon$ (respectively $y > x + \varepsilon$) $s(y) = y$.

It is due essentially to Mather [9] and Segal [11] that for pseudogroups Γ which are germ-connected to the identity, the maps $BH_l \to B\Gamma$ and $BK_l \to \Omega B\Gamma$ are homology equivalences. Further, we have a corollary from Section 3 of this paper.

1.11. Corollary. *If Γ is germ connected to the identity, and if $BG \to B\Gamma$ is null-homotopic, then $B\Gamma^1$ is weakly equivalent to $B\Gamma * B\Gamma \vee SB\Gamma \vee S^2BG$ and $B\Gamma^2$ is weakly equivalent to $B\Gamma^1 * B\Gamma^1 \vee SB\Gamma^1 \vee SB\Gamma * B\Gamma \vee 2S^2B\Gamma$.*

1.12. Example. Let Γ be the pseudogroup of orientation preserving C^r diffeomorphisms between open subsets of the reals, $1 \leq r \leq \infty$. Then Γ is germ-connected to the identity, and Tsuboi [13] has shown that the maps $BG \to B\Gamma$ are nullhomotopic, so the corollary applies.

1.13. Tools and notation. We work with the homotopy direct limit (hocolim) of diagrams of topological spaces. The underlying graphs of our diagrams have a finite number of vertices, and no cycles. The hocolim is constructed as follows, in stages. The 0th stage is the disjoint union of the spaces in the diagram. The 1st stage is constructed by gluing in the mapping cylinders of all of the maps in the diagram, the second stage glues in mapping triangles, that is, copies of $\Delta^2 \times A$ for each sequence of maps $A \to B \to C$ in the diagram, and so on. If D is a diagram its hocolim is denoted $|D|$.

A map between diagrams will be always between diagrams whose underlying graphs are identical, and is a collection of maps between the corresponding spaces in the respective diagrams, so that all appropriate diagrams commute. In this way there are diagrams of diagrams, and an important tool is the *Fubini theorem*, stating that one can take the hocolims in either order. See [12, 2] for solid exposition

If a map between diagrams is composed of homotopy equivalences, the hocolims are homotopy equivalent If the map between diagrams is an isomorphism in homology for each space in the diagrams, the map between the hocolims is an isomorphism in homology, because there is a "Mayer–Vietoris" spectral sequence for calculating the homology of a hocolim. A van Kampen theorem holds for hocolims, so that one can sometimes prove, using the Whitehead theorem, that a map between diagrams which is an isomorphism in homology for each space in the diagrams induces a homotopy equivalence between the hocolims

It is harder to prove that a map between diagrams induces a nullhomotopic map between the hocolims Of course, it does not suffice that each map involved is nullhomotopic We restrict this difficulty to Proposition 3 6

To illustrate the use of the Fubini theorem, we prove the following well-known fact

1.14. Lemma. *Let A, B, C be pointed spaces. Then $A * (B \vee C)$ is homotopy equivalent to $A * B \vee A * C$*

Proof. If X and Y are pointed spaces, $X * Y$ is the pushout of the diagram $X \leftarrow X \times Y \rightarrow Y$ Now consider the diagram

$$\begin{array}{ccccc} A & \leftarrow & A \times B & \rightarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ A & \leftarrow & A \times * & \rightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A & \leftarrow & A \times C & \rightarrow & C \end{array}$$

Realizing the columns first, we get the pushout $A \leftarrow A \times (B \vee C) \rightarrow B \vee C$, and realizing the rows first, we obtain the pushout $A * B \leftarrow * \rightarrow A * C$ $\square$

1.15. Organization. In Section 2, we arrange some results from [3] in a useful form Section 3 constructs a certain cofibration and pushout, and Section 4 contains the proof of Theorem 1 4 and Corollary 1 11

2. Extensions with isolated singularities

We recall a definition and result from [3]

2.1. Definition. Let $\Gamma \subseteq \Gamma^p$ be pseudogroups of orientation preserving homeomorphisms between open subsets of $\mathbb{R}$, so that there is some $B \subset \mathbb{R}$ such that

(i) $\Gamma(B) = \Gamma^p(B)$,

(ii) if $g: U \to V$ is in Γ^p, there exists $S \subset U \cap \Gamma(B)$ with $L'(S) = \emptyset$ so that for each component I of $U - S$ there exists $g_I \in \Gamma$ with $g|_I \equiv g_I$.

Then Γ^p is an *extension of Γ with isolated singularities in the orbit of B.*

2.2. Remark. The inclusions $\Gamma^n \subseteq \Gamma^{n+1}$, $n \geq 0$, are evidently extensions with isolated singularities in the orbit of the basepoint B

Let $\Gamma \subseteq \Gamma^p$ be an extension with isolated singularities in the orbit of B, and let $G \subseteq G^p$ be the groups of germs of the pseudogroups at the point B.

2.3. Theorem [3, 1.4]. *There is a weak equivalence from the hocolim of the diagram*

$$\begin{tikzcd}
 & & {|G^p\backslash\mathbb{R}|} & & \\
{|G^p\backslash\mathbb{R}_l|} \arrow[urr] \arrow[drr, "f_l"'] & {|G\backslash\mathbb{R}_l|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {|G\backslash\mathbb{R}|} \arrow[u] \arrow[d, "p"] & {|G\backslash\mathbb{R}_r|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {|G^p\backslash\mathbb{R}_r|} \arrow[ull] \arrow[dll, "f_r"] \\
 & & B\Gamma & &
\end{tikzcd} \tag{1}$$

to $B\Gamma^p$

The spaces and maps in (1) require explanation. Although G and G^p are just groups of germs, Tsuboi [13] constructs a foliated $\mathbb{R}$-bundle $|G^p\backslash\mathbb{R}|$ over BG^p which has $BG^p \times B$ as a leaf, whose holonomy is exactly that specified by G^p. Restricting to G, $\mathbb{R}_r = \{x \in \mathbb{R} : x > B\}$, and $\mathbb{R}_l = \{x \in \mathbb{R} : x < B\}$ we obtain the other unidentified spaces in (1). The map p classifies the foliated bundle, and f_l and f_r exist because the "singularity" exists only at B. See [3] for details.

We will work with a slight modification of (1). Namely, there exists a closed (not saturated) neighborhood of $BG^p \times B$ in $|G^p\backslash\mathbb{R}|$ which can be identified with $BG^p \times I$, $I = [-1, 1]$, where $BG^p \times B$ is identified with $BG^p \times 0$ in $BG^p \times I$. We call this neighborhood $|G^p\backslash I|$ (abusing notation) and its restriction over BG is $|G\backslash I|$. Then we will write points in $|G^p\backslash I|$ as (x, t), $x \in BG^p$, $t \in I$ and we set $|G^p\backslash 1| = \{(x, 1) : x \in BG^p\}$, and similarly define $|G^p\backslash -1|, |G\backslash 1|, |G\backslash -1|$. Inclusions give a diagram $\mathscr{D}$

$$\begin{tikzcd}
 & & {|G^p\backslash I|} & & \\
{|G^p\backslash -1|} \arrow[urr] \arrow[drr, "f_{-1}"'] & {|G\backslash -1|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {|G\backslash I|} \arrow[u] \arrow[d, "p"] & {|G\backslash 1|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {|G^p\backslash 1|} \arrow[ull] \arrow[dll, "f_1"] \\
 & & B\Gamma & &
\end{tikzcd} \tag{$\mathscr{D}$}$$

which maps by inclusion on each factor to the diagram (1). On each space in $\mathscr{D}$, the inclusion to (1) is a homotopy equivalence; hence $|\mathscr{D}|$ is weakly homotopy equivalent to $B\Gamma^p$. We will use $\mathscr{D}$.

3. Cofibration and pushout

In this section, $\Gamma \subseteq \Gamma^p$ is an extension with isolated singularities in the orbit of B (see Definition 2.1).

Consider the diagram $\mathscr{X}$:

$$(\mathscr{X}) \qquad \begin{tikzcd}
 & & {|G^p\backslash I|} & & \\
{|G^p\backslash -1|} \arrow[urr] \arrow[drr, "f_{-1}"'] & -1 \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & I \arrow[u] \arrow[d] & 1 \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {|G^p\backslash 1|} \arrow[ull] \arrow[dll, "f_1"] \\
 & & B\Gamma & &
\end{tikzcd}$$

where I is the fiber over the basepoint of BG and BG^p, and $1, -1$ pertain to this fiber. Thus, the maps in $\mathscr{X}$ are restrictions of maps in $\mathscr{D}$, so we have a map of diagrams $\mathscr{X} \to \mathscr{D}$. Also, $\mathscr{X}$ maps to the diagram $*$:

$$(*) \qquad \begin{tikzcd}
 & & * & & \\
* \arrow[urr] \arrow[drr] & * \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & * \arrow[u] \arrow[d] & * \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & * \arrow[ull] \arrow[dll] \\
 & & * & &
\end{tikzcd}$$

in the only possible way, and we calculate the homotopy cofiber of $|\mathscr{X}| \to |\mathscr{D}|$ as the diagram of pushouts $\mathscr{C}$:

$$(\mathscr{C}) \qquad \begin{tikzcd}[column sep=small]
 & & {\big|* \leftarrow |G^p\backslash I| \rightarrow |G^p\backslash I|\big|} & & \\
{\big|* \leftarrow |G^p\backslash -1| \rightarrow |G^p\backslash -1|\big|} \arrow[urr] \arrow[drr] & {\big|* \leftarrow -1 \rightarrow |G\backslash -1|\big|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {\big|* \leftarrow I \rightarrow |G\backslash I|\big|} \arrow[u] \arrow[d] & {\big|* \leftarrow 1 \rightarrow |G\backslash 1|\big|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {\big|* \leftarrow |G^p\backslash 1| \rightarrow |G^p\backslash 1|\big|} \arrow[ull] \arrow[dll] \\
 & & {|* \leftarrow B\Gamma \rightarrow B\Gamma|} & &
\end{tikzcd}$$

Now, consider the diagram $\mathscr{B}$

$$(\mathscr{B}) \qquad \begin{tikzcd} & & * & & \\ * \arrow[urr] \arrow[drr] & {|G\backslash -1|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {|G\backslash I|} \arrow[u] \arrow[d] & {|G\backslash 1|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & * \arrow[ull] \arrow[dll] \\ & & * & & \end{tikzcd}$$

and note that we have a map of diagrams $\mathscr{C} \to \mathscr{B}$; to define the map $|* \leftarrow I \to |G\backslash I|| \to |G\backslash I|$, send the mapping cylinder of $* \leftarrow I$ to a contraction of I to 0. Further, it is easy to check that for each space in the diagram $\mathscr{C}$, the map is a homotopy equivalence. We have

3.1. Proposition. *Up to weak homotopy, there is a cofibration $|\mathscr{X}| \to B\Gamma^p \to S^2BG$*

Proof. It suffices to observe that $|\mathscr{B}|$ is homotopy equivalent to S^2BG, the hocolim of the middle row of $\mathscr{B}$ is homotopy equivalent to SBG, and therefore $|\mathscr{B}|$ is homotopy equivalent to $|* \leftarrow SBG \to *|$, that is to S^2BG

Before proceeding, let us briefly discuss the geometry of the map $B\Gamma^p \to S^2BG$ given in Proposition 3 1 (see [3, Section 4]) If M is a compact n-manifold with a codimension-1 Γ^p-foliation, then one can discard a finite number of compact leaves K_j, and a finite number of compact $(n-1)$-submanifolds N_i with boundary W_i of noncompact leaves L_i, to obtain an open manifold with a Γ-foliation (see Fig 2) On $L_i - N_i$ the holonomy is nonsingular, in particular, $\pi_1 W_i \to G^p$ factors through G Thus, a small neighborhood $U_i \approx W_i \times D^2$ of W_i maps to $BG \times D^2$. Letting

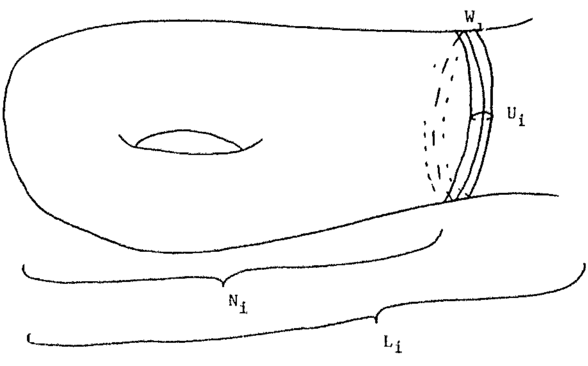

Fig 2

$U = \bigcup U_i$, we have a composition $M \to M/M - U \to S^2BG$, where $M/M - U$ means that $M - U$ is collapsed to a point. We see that the map $B\Gamma^p \to S^2BG$ is a sort of Thom map.

We now find a condition under which the cofibration of Proposition 3.1 splits. Note that it does not split for the pair $\Gamma \subset \Gamma^1$ of Basic example 1.9, where it takes the form $S^3 \vee S^2 \to S^3 \to S^3$.

Let $\mathcal{Y}$ be the diagram

$$(\mathcal{Y}) \qquad \begin{tikzcd} & & {|G\backslash I|} & & \\ {|G\backslash -1|} \arrow[urr] \arrow[drr] & -1 \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & I \arrow[u] \arrow[d] & 1 \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {|G\backslash 1|} \arrow[ull] \arrow[dll] \\ & & {|G\backslash I|} & & \end{tikzcd}$$

where, as before, I is the fiber over the basepoint of BG and $1, -1$ are the endpoints of I. Let $\mathcal{V}$ be the diagram

$$(\mathcal{V}) \qquad \begin{tikzcd} & & {|G\backslash I|} & & \\ {|G\backslash -1|} \arrow[urr] \arrow[drr] & {|G\backslash -1|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {|G\backslash I|} \arrow[u] \arrow[d] & {|G\backslash 1|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {|G\backslash 1|} \arrow[ull] \arrow[dll] \\ & & {|G\backslash I|} & & \end{tikzcd}$$

whose maps are the identity or the obvious inclusions. With inclusions, there is a map of diagrams $\mathcal{Y} \to \mathcal{V}$. Also, $\mathcal{Y}$ maps to $\mathcal{X}$, using inclusions for every space in $\mathcal{Y}$ except for the bottom $|G\backslash I|$, which maps to the $B\Gamma$ in $\mathcal{X}$ using the map p from the diagram $\mathcal{D}$. Again, we use the Fubini theorem to calculate the pushout of $|\mathcal{V}| \leftarrow |\mathcal{Y}| \to |\mathcal{X}|$ as the hocolim of the following diagram of pushouts $\mathcal{W}$:

$$(\mathcal{W}) \qquad \begin{tikzcd}[column sep=small] & & {\big||G\backslash I| \leftarrow |G\backslash I| \to |G'\backslash I|\big|} & & \\ {\big||G\backslash -1| \leftarrow |G\backslash -1| \to |G'\backslash -1|\big|} \arrow[urr] \arrow[drr] & {\big||G\backslash -1| \leftarrow -1 \to -1\big|} \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & {\big||G\backslash I| \leftarrow I \to I\big|} \arrow[u] \arrow[d] & {\big||G\backslash 1| \leftarrow 1 \to 1\big|} \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & {\big||G\backslash 1| \leftarrow |G\backslash 1| \to |G'\backslash 1|\big|} \arrow[ull] \arrow[dll] \\ & & {\big||G\backslash I| \leftarrow |G\backslash I| \to B\Gamma\big|} & & \end{tikzcd}$$

Now, $\mathscr{W}$ maps to $\mathscr{D}$ in the (more or less) obvious way, and the map is a homotopy equivalence for each pushout in $\mathscr{W}$, so that $|\mathscr{W}| \to |\mathscr{D}|$ is a weak homotopy equivalence

3.2. Proposition. *$B\Gamma^{p}$ is weakly homotopy equivalent to the pushout $D^2 \times BG \leftarrow (S^1 \times BG) \cup_{S^1} (D^2) \to |\mathscr{X}|$*

Proof. Certainly $|\mathscr{V}|$ is homotopy equivalent to $D^2 \times BG$, we need to check that $|\mathscr{Y}|$ is homotopy equivalent to $(S^1 \times BG) \cup_{S^1} (D^2)$, and that $|\mathscr{Y}| \to |\mathscr{V}|$ corresponds to $(S^1 \times BG) \cup_{S^1} (D^2) \to BG \times D^2$. But considering that the perimeter of $\mathscr{Y}$

$$\begin{tikzcd}
& {|G\backslash I|} & \\
{|G\backslash -1|} \arrow[ur] \arrow[dr] & & {|G\backslash 1|} \arrow[ul] \arrow[dl] \\
& {|G\backslash I|} &
\end{tikzcd}$$

is homotopy equivalent to $S^1 \times BG$, this is evident □

3.3. Lemma. *If $A \to B$ is nullhomotopic, then the pushout $C \leftarrow A \to B$ is weakly equivalent to $C/A \vee B$, where C/A is the homotopy cofiber of $A \to C$*

The proof is left to the reader

3.4. Corollary. *If $|\mathscr{Y}| \to |\mathscr{X}|$ is nullhomotopic, $B\Gamma^{p}$ is weakly equivalent to $S^2 BG \vee |\mathscr{X}|$*

In Basic example 1 9, the pushout of Proposition 3 2 takes the form $S^2 \leftarrow (S^1 \times S^2)/S^1 \to S^3 \vee S^2$ The map $(S^1 \times S^1)/S^1 \to S^3 \vee S^2$ induces an isomorphism in H_2, and Lemma 3 3 cannot be applied

We will now introduce an auxiliary diagram and maps, which will provide us with a condition under which $|\mathscr{Y}| \to |\mathscr{X}|$ is nullhomotopic Let $\mathscr{Z}$ be the diagram

$$(\mathscr{Z}) \qquad \begin{tikzcd}
& & B\Gamma \times B\Gamma & & \\
B\Gamma \times B\Gamma \arrow[urr, "\mathrm{id}"] \arrow[drr, "\pi_l"'] & -1 \arrow[l] \arrow[r] \arrow[ur] \arrow[dr] & I \arrow[u] \arrow[d] & 1 \arrow[l] \arrow[r] \arrow[ul] \arrow[dl] & B\Gamma \times B\Gamma \arrow[ull, "\mathrm{id}"'] \arrow[dll, "\pi_r"] \\
& & B\Gamma & &
\end{tikzcd}$$

where the π_l and π_r are projections to the left and right factor, respectively, id is the identity, and the map $I \to B\Gamma$ is the composition $I \to |G\backslash I| \to B\Gamma$ The map $I \to B\Gamma \times B\Gamma$ sends t to (p, q) for all t, where p (respectively q) is the image of -1 (respectively 1) in the map $I \to B\Gamma$

We now construct a map of diagrams $\mathscr{X} \to \mathscr{Y}$ For the -1, 1, I, and $B\Gamma$ of $\mathscr{X}$, the map is the identity For the $|G^P\backslash I|$, $|G^P\backslash -1|$, and $|G^P\backslash 1|$ we use the following map $F\ |G^P\backslash I| \to B\Gamma \times B\Gamma$, given by $F(x, t) = (f_{-1}(x, -1), f_1(x, 1))$ It is straightforward to check that all diagrams commute, so that the map $\mathscr{X} \to \mathscr{Z}$ is well defined

3.5. Proposition. *$|\mathscr{Z}|$ is homotopy equivalent to $B\Gamma * B\Gamma \vee SB\Gamma$*

Proof. The middle row of $\mathscr{Z}$ has hocolim $2(B\Gamma \times B\Gamma)$. So it suffices to show that for any space X, the pushout (hocolim of the diagram) $X \leftarrow X \times X \vee X \times X \to X \times X$, where the left map is $\pi_l \vee \pi_r$, the wedge of the two projections, and the right map is the fold, is homotopy equivalent to $SX \vee X \times X$.

Consider the diagram

$$\begin{tikzcd}
X & {*} \arrow[l] \arrow[r] & {*} \\
X \amalg X \arrow[u] \arrow[d] & {*} \amalg {*} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & {*} \arrow[u] \arrow[d] \\
X \amalg X & X \times X \amalg X \times X \arrow[l, "\pi_l \amalg \pi_r"'] \arrow[r, "\mathrm{id} \amalg \mathrm{id}"] & X \times X
\end{tikzcd}$$

Evaluating first the pushouts of the columns, we obtain $|X \leftarrow X \times X \amalg X \times X \to X \times X|$ Evaluating the pushouts of the rows, we get $|X \leftarrow X \vee X \to X * X|$ Since $X \vee X \to X * X$ is nullhomotopic, the latter is homotopy equivalent, by Lemma 3 3, to $SX \vee X * X$ $\square$

3.6. Proposition. *If $BG \to B\Gamma$ is nullhomotopic, and $|\mathscr{X}| \to |\mathscr{Z}|$ is a weak equivalence, then $B\Gamma^P$ is weakly equivalent to $B\Gamma * B\Gamma \vee SB\Gamma \vee S^2BG$*

Proof. By Corollary 3 4 and Proposition 3 5, it suffices to prove that if $BG \to B\Gamma$ is nullhomotopic, then $|\mathscr{Y}| \to |\mathscr{Z}|$ is nullhomotopic. We will construct a nullhomotopy Because I is a deformation retract of a neighborhood in $|G\backslash I|$ (see the construction in [13] or [3]), and since $p\ |G\backslash I| \to B\Gamma$ is nullhomotopic, there exists a map $H\ |G\backslash I| \times [0, 1] \to B\Gamma$ so that $H((x, t), 0) = p(x, t)$, $H((*, t), s) = p(*, t)$, and $H((x, t), 1) = p(*, t)$ Now define a mapping $E\ |G\backslash I| \times [0, 1] \to B\Gamma \times B\Gamma$ by $E((x, t), s) = (H((x, -1), s), H((x, 1), s))$ Call the restriction of E to $|G\backslash -1| \times [0, 1]$ and $|G\backslash 1| \times [0, 1]$, E_{-1} and E_1 respectively Using H on the bottom $|G\backslash I|$ in $\mathscr{Y}$, E_{-1}, E_1 and E on the $|G\backslash -1|$, $|G\backslash 1|$ and $|G\backslash I|$ respectively, and the identity on the -1, 1, and I, one can check that we have a homotopy of maps from $\mathscr{Y}$ to $\mathscr{Z}$ For $s = 0$, the map is the map already defined, and for $s = 1$, the map has contractible image in $|\mathscr{Z}|$ $\square$

4. Proof of Theorem 1.4

We will need the following two results, whose proofs follow at the end of the section Recall the notation from Section 1

4.1. Proposition. *If Γ is germ-connected to the identity, then the map $F: BG^1 \to B\Gamma \times B\Gamma$ induces isomorphism in homology*

4.2. Proposition. *Let Γ be a pseudogroup satisfying Assumptions 1 1. Let G^n be the group of germs at B of the pseudogroup $B\Gamma$. Then $BG^n \to B\Gamma^n$ is nullhomotopic, $n \geq 1$.*

4.3. Corollary. *If Γ is germ-connected to the identity, the map $|\mathscr{X}| \to |\mathscr{Y}|$ of Section 3 is a homotopy equivalence.*

Proof. By Proposition 4 1, $|\mathscr{X}| \to |\mathscr{Z}|$ induces an isomorphism in homology By Corollary 1.6 and the van Kampen theorem, both spaces are simply connected. The result follows from Whitehead's theorem □

Proof of Corollary 1.11. This now follows from Proposition 3.6 □

Now, for $n \geq 0$, let $\mathscr{X}^n, \mathscr{Y}^n, \mathscr{Z}^n$ be the diagrams $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ of Section 3, with Γ^n, G^n, G^{n+1} in place of Γ, G, G^p

4.4. Corollary. $|\mathscr{X}^n| \to |\mathscr{Z}^n|$ *is a homotopy equivalence for $n \geq 1$*

Proof. This follows directly from Proposition 4 1, since Assumptions 1.1 guarantee that the Γ^n are germ-connected to the identity, $n \geq 1$ □

4.5. Corollary. $B\Gamma^n \to B\Gamma^{n+1}$ *is nullhomotopic, $n \geq 1$*

Proof. By Theorem 2 3 it suffices to prove that $B\Gamma^n$ is contractible in $|\mathscr{X}^n|$, and hence, by Corollary 4.4, that $B\Gamma^n$ is contractible in $|\mathscr{Z}^n|$ This follows from Proposition 3 5 □

Proof of Theorem 1.4. Theorem 1 4(i) is Corollary 4 5 If $n \geq 1$, $BG^n \to B\Gamma^n$ is nullhomotopic by Proposition 4 2, so by Proposition 3 6 $|\mathscr{Y}^n| \to |\mathscr{Z}^n|$ is nullhomotopic Thus by Corollary 4 3, $B\Gamma^{n+1}$ is weakly equivalent to $B\Gamma^n * B\Gamma^n \vee SB\Gamma^n \vee S^2BG^n$, $n = 1$ But the suspension of a map which induces isomorphism in homology is a homotopy equivalence, so if $n \geq 2$, by Proposition 4 1, $B\Gamma^{n+1}$ is weakly equivalent to $B\Gamma^n * B\Gamma^n \vee SB\Gamma^n \vee S^2(B\Gamma^{n-1} \times B\Gamma^{n-1})$. Now suspension splits products, whence follows the theorem □

Proof of Proposition 4.1. Write $G^{l} = H^{l} \times H^{r}$, where H^{l} (respectively H^{r}) is the subgroup of G^{l} whose elements have support to the left (respectively right) of B Let D^{l} be the group of Γ homeomorphisms from $(-\infty, B)$ to itself. Note that restriction to a neighborhood to the left of B gives a surjection $D^{l} \to H^{l}$, whose kernel is acyclic by [11, 3 2] (The results of [11] follow for pseudogroups which are germ-connected to the identity) Thus $BD^{l} \to BH^{l}$ is an isomorphism in homology Further, the map $BD^{l} \to B\Gamma$ is a homology isomorphism by [11, 3.1] The diagram

$$\begin{tikzcd} & B\Gamma & \\ BD^{l} \arrow[ur] \arrow[dr] & & {|H^{l}\backslash -1|} \arrow[ul] \arrow[dl] \\ & BH^{l} & \end{tikzcd}$$

commutes up to homotopy Thus $|H^{l}\backslash -1| \to B\Gamma$ is an isomorphism in homology, and the same holds for H^{r} □

Proof of Proposition 4.2. We briefly repeat an elementary version of Tsuboi's argument in [13] Let $H^{n\,l}$ be as above, but for the group G^{n} The diagram, where $G^{n} \to H^{n\,l}$ is projection,

$$\begin{tikzcd} BG^{n} \arrow[rr] \arrow[dr] & & BH^{n,l} \arrow[dl] \\ & B\Gamma^{n} & \end{tikzcd}$$

is homotopy commutative, because the Γ^{n} structure on $|G^{n}\backslash -1|$ is the same as that induced by factoring through $BH^{n\,l}$ The map $BH^{n\,l} \to B\Gamma^{n}$ is nullhomotopic because, in $|H^{n\,l}\backslash I|$, the section $BH^{n\,l} \times \frac{1}{2}$ pulls back the trivial Γ^{n} structure Thus $BG^{n} \to B\Gamma^{n}$ is nullhomotopic, $n \geq 1$ □

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